CAPÍTULO

2

Métodos de Integración

[1]

2.8 Combinación de métodos de integración

2.8.1 Introducción

En las secciones anteriores hemos tratado con tres métodos de integración: cambio de variable, por partes y fracciones parciales y algunas técnicas de integración que hacen uso de identidades trigonométricas.

Mediante los métodos y técnicas de integración hemos aprendido a calcular familias de integrales:

$$\int x^n e^{ax}\,dx; \qquad \int x^n \operatorname{sen} ax\,dx; \qquad \int x^n \cos ax\,dx;$$
$$\int x^r \ln ax\,dx; \qquad \int \operatorname{sen}^r x \cos x\,dx; \qquad \int \cos^r x \operatorname{sen} x\,dx;$$
$$\int \tan^r x \sec^2 x\,dx; \qquad \int e^{ax} \operatorname{sen} kx\,dx; \qquad \int e^{ax} \cos kx\,dx;$$

entre otras. En esta sección trataremos con integrales que no son del estilo de las ya tratadas, pero que pueden ser llevadas a estas después de haber aplicado un cambio de variable adecuado, o bien después de haber integrado por partes.

2.8.2 Cambio de variable y luego integración por partes

Integrales de la forma

$$\int e^{\sqrt[n]{ax+b}}\,dx; \qquad \int \operatorname{sen} \sqrt[n]{ax+b}\,dx;$$
$$\int \cos \sqrt[n]{ax+b}\,dx; \qquad \int \ln \sqrt[n]{ax+b}\,dx \qquad \text{donde } n \in \mathbb{N} \;\&\; n \geq 2.$$

1. canek.azc.uam.mx: 13/ 1/ 2017

Al aplicar un cambio de variable apropiado, la integral original se convierte en otra integral en donde el integrando no contiene dicha raíz.

$$\sqrt[n]{ax+b} = y \;\Rightarrow\; ax+b = y^n \;\Rightarrow\; x = \frac{1}{a}(y^n - b) \quad \& \quad dx = \frac{n}{a}y^{n-1}\,dy.$$

Luego,

- $\displaystyle\int e^{\sqrt[n]{ax+b}}\,dx = \int e^y\left(\frac{n}{a}y^{n-1}\,dy\right) = \frac{n}{a}\int y^{n-1}e^y\,dy.$
- $\displaystyle\int \operatorname{sen}\sqrt[n]{ax+b}\,dx = \int (\operatorname{sen} y)\left(\frac{n}{a}y^{n-1}\,dy\right) = \frac{n}{a}\int y^{n-1}\operatorname{sen} y\,dy.$
- $\displaystyle\int \cos\sqrt[n]{ax+b}\,dx = \int (\cos y)\left(\frac{n}{a}y^{n-1}\,dy\right) = \frac{n}{a}\int y^{n-1}\cos y\,dy.$

Obtenemos aquí integrales que se calculan aplicando el método de integración por partes $n-1$ veces. Es decir, después de un cambio de variable adecuado, hemos obtenido integrales pertenecientes a grandes familias que se calculan mediante integración por partes.

Ejemplo 2.8.1 *Calcular la integral* $\displaystyle\int e^{\sqrt{2x-3}}\,dx$.

▼ Si $\sqrt{2x-3} = y$, entonces $2x-3 = y^2 \;\Rightarrow\; x = \frac{1}{2}\left(y^2+3\right)$ & $dx = y\,dy$.

Luego,

$$\int e^{\sqrt{2x-3}}\,dx = \int e^y y\,dy = \int ye^y\,dy.$$

Aplicamos integración por partes, tomando

$$u = y \;\&\; dv = e^y\,dy \;\Rightarrow\; du = dy \;\&\; v = e^y.$$

Entonces,

$$\int e^{\sqrt{2x-3}}\,dx = \int ye^y\,dy = ye^y - \int e^y\,dy = ye^y - e^y + C = (y-1)e^y + C = \left(\sqrt{2x-3}-1\right)e^{\sqrt{2x-3}} + C.$$

□

Ejemplo 2.8.2 *Calcular la integral* $\displaystyle\int \cos\sqrt[3]{x}\,dx$.

▼ Si $\sqrt[3]{x} = y$, entonces $x = y^3$ & $dx = 3y^2\,dy$. Luego,

$$\int \cos\sqrt[3]{x}\,dx = \int (\cos y)3y^2\,dy = 3\int y^2\cos y\,dy.$$

Aplicamos integración por partes, seleccionando

$$u = y^2 \;\&\; dv = \cos y\,dy \;\Rightarrow\; du = 2y\,dy \;\&\; v = \operatorname{sen} y.$$

Entonces:

$$\int y^2\cos y\,dy = y^2\operatorname{sen} y - \int (\operatorname{sen} y)2y\,dy = y^2\operatorname{sen} y - 2\int y\operatorname{sen} y\,dy.$$

De nuevo por partes:

$$\hat{u} = y \;\&\; d\hat{v} = \operatorname{sen} y \;\Rightarrow\; d\hat{u} = dy \;\&\; v = -\cos y.$$

Obtenemos:

$$\int y^2 \cos y \,dy = y^2 \operatorname{sen} y - 2\int y \operatorname{sen} y \,dy = y^2 \operatorname{sen} y - 2\left[-y\cos y + \int \cos y \,dy\right] =$$
$$= y^2 \operatorname{sen} y + 2y\cos y - 2\operatorname{sen} y + C.$$

Por lo tanto,

$$\int \cos \sqrt[3]{x}\,dx = 3\int y^2\cos y\,dy = 3\left[y^2 \operatorname{sen} y + 2y\cos y - 2\operatorname{sen} y\right] + C = 3\left(y^2-2\right)\operatorname{sen} y + 6y\cos y + C =$$
$$= 3\left[\left(\sqrt[3]{x}\right)^2 - 2\right]\operatorname{sen}\sqrt[3]{x} + 6\sqrt[3]{x}\cos\sqrt[3]{x} + C = 3\left[\sqrt[3]{x^2} - 2\right]\operatorname{sen}\sqrt[3]{x} + 6\sqrt[3]{x}\cos\sqrt[3]{x} + C.$$

□

2.8.3 Integración por partes y luego otra técnica

Integrales de la forma

$$\int x^n \arcsen x\,dx; \quad \int x^n \arccos x\,dx; \quad \int x^n \arctan x\,dx;$$
$$\int x^n \operatorname{arccot} x\,dx; \quad \int x^n \operatorname{arcsec} x\,dx; \quad \int x^n \operatorname{arccsc} x\,dx,$$

donde n es un entero no-negativo.

Aquí conviene aplicar integración por partes seleccionando

$$dv = x^n\,dx \;\&\; u = \text{arc}\_\_.$$

- $\int x^n \operatorname{arcsen} x\,dx$.

$$\underbrace{\int x^n \operatorname{arcsen} x\,dx}_{\begin{array}{|lcl|} \hline u = \operatorname{arcsen} x & \& & dv = x^n\,dx; \\ du = \dfrac{dx}{\sqrt{1-x^2}} & \& & v = \dfrac{x^{n+1}}{n+1}. \\ \hline \end{array}} = \frac{x^{n+1}}{n+1}\operatorname{arcsen} x - \int \frac{x^{n+1}}{n+1}\frac{dx}{\sqrt{1-x^2}} =$$
$$= \frac{x^{n+1}}{n+1}\operatorname{arcsen} x - \frac{1}{n+1}\int \frac{x^{n+1}}{\sqrt{1-x^2}}\,dx.$$

Ahora, por sustitución trigonométrica:

$$x = \operatorname{sen}\theta \;\Rightarrow\; dx = \cos\theta\,d\theta \quad \& \quad \sqrt{1-x^2} = \sqrt{1-\operatorname{sen}^2\theta} = \cos\theta.$$

Luego,

$$\int x^n \operatorname{arcsen} x\,dx = \frac{x^{n+1}}{n+1}\operatorname{arcsen} x - \frac{1}{n+1}\int \frac{x^{n+1}}{\sqrt{1-x^2}}\,dx =$$
$$= \frac{x^{n+1}}{n+1}\operatorname{arcsen} x - \frac{1}{n+1}\int \frac{\operatorname{sen}^{n+1}\theta}{\cos\theta}(\cos\theta\,d\theta) =$$
$$= \frac{x^{n+1}}{n+1}\operatorname{arcsen} x - \frac{1}{n+1}\int \operatorname{sen}^{n+1}\theta\,d\theta,$$

donde $n+1$ es un entero positivo que puede ser par o impar.

- $\displaystyle\int x^n \operatorname{arccos} x \, dx$.

$$\int x^n \arccos x \, dx = \frac{x^{n+1}}{n+1} \arccos x + \frac{1}{n+1} \int \operatorname{sen}^{n+1}\theta \, d\theta.$$

Se obtiene de manera análoga al caso anterior.

- $\displaystyle\int x^n \arctan x \, dx$.

$$\underbrace{\int x^n \arctan x \, dx}_{\boxed{\begin{array}{ccc} u = \arctan x & \& & dv = x^n \, dx; \\ du = \dfrac{dx}{1+x^2} & \& & v = \dfrac{x^{n+1}}{n+1}. \end{array}}} = \frac{x^{n+1}}{n+1} \arctan x - \int \frac{x^{n+1}}{n+1} \frac{dx}{1+x^2} =$$

$$= \frac{x^{n+1}}{n+1} \arctan x - \frac{1}{n+1} \int \frac{x^{n+1}}{x^2+1} \, dx,$$

donde $n+1$ es un número natural y donde $\dfrac{x^{n+1}}{x^2+1}$ es una división de polinomios.

- $\displaystyle\int x^n \operatorname{arccot} x \, dx$.

$$\int x^n \operatorname{arccot} x \, dx = \frac{x^{n+1}}{n+1} \operatorname{arccot} x + \frac{1}{n+1} \int \frac{x^{n+1}}{x^2+1} \, dx.$$

Se obtiene de manera análoga al último caso.

- $\displaystyle\int x^n \operatorname{arcsec} x \, dx$.

$$\underbrace{\int x^n \operatorname{arcsec} x \, dx}_{\boxed{\begin{array}{ccc} u = \operatorname{arcsec} x & \& & dv = x^n \, dx; \\ du = \dfrac{dx}{x\sqrt{x^2-1}} & \& & v = \dfrac{x^{n+1}}{n+1}. \end{array}}} = \frac{x^{n+1}}{n+1} \operatorname{arcsec} x - \int \frac{x^{n+1}}{n+1} \frac{dx}{x\sqrt{x^2-1}} =$$

$$= \frac{x^{n+1}}{n+1} \operatorname{arcsec} x - \frac{1}{n+1} \int \frac{x^n}{\sqrt{x^2-1}} \, dx.$$

Ahora, por sustitución trigonométrica:

$$x = \sec\theta \;\Rightarrow\; dx = \sec\theta \cdot \tan\theta \; d\theta \quad \& \quad \sqrt{x^2-1} = \sqrt{\sec^2\theta - 1} = \tan\theta.$$

Luego,

$$\begin{aligned} \int x^n \operatorname{arcsec} x \, dx &= \frac{x^{n+1}}{n+1} \operatorname{arcsec} x - \frac{1}{n+1} \int \frac{\sec^n\theta}{\tan\theta} (\sec\theta \cdot \tan\theta) \, d\theta = \\ &= \frac{x^{n+1}}{n+1} \operatorname{arcsec} x - \frac{1}{n+1} \int \sec^{n+1}\theta \, d\theta, \end{aligned}$$

donde $n+1$ es un natural, que puede ser par o impar.

- $\displaystyle\int x^n \operatorname{arccsc} x \, dx$.

$$\int x^n \operatorname{arccsc} x \, dx = \frac{x^{n+1}}{n+1} \operatorname{arccsc} x + \frac{1}{n+1} \int \sec^{n+1}\theta \, d\theta.$$

Se obtiene de manera análoga al último caso.

Ejemplo 2.8.3 *Calcular la integral* $\int x^2 \operatorname{arcsen} x\, dx$.

▼ Primero aplicamos la integración por partes

$$\underbrace{\int x^2 \operatorname{arcsen} x\, dx}_{\begin{array}{|ccc|} \hline u = \operatorname{arcsen} x & \& & dv = x^2\,dx; \\ du = \dfrac{dx}{\sqrt{1-x^2}} & \& & v = \dfrac{x^3}{3}. \\ \hline \end{array}} = (\operatorname{arcsen} x)\frac{x^3}{3} - \int \frac{x^3}{3}\frac{dx}{\sqrt{1-x^2}} = $$

$$= \frac{x^3}{3}\operatorname{arcsen} x - \frac{1}{3}\int \frac{x^3}{\sqrt{1-x^2}}\,dx.$$

Ahora, por sustitución trigonométrica:

$$x = \operatorname{sen}\theta \;\Rightarrow\; dx = \cos\theta\, d\theta \;\Rightarrow\; \sqrt{1-x^2} = \sqrt{1-\operatorname{sen}^2\theta}\,dx = \cos\theta.$$

Luego,

$$\int \frac{x^3}{\sqrt{1-x^2}}\,dx = \int \frac{\operatorname{sen}^3\theta}{\cos\theta}(\cos\theta)\,d\theta = \int \operatorname{sen}^3\theta\,d\theta = \int \operatorname{sen}^2\theta\ \operatorname{sen}\theta\,d\theta =$$

$$= \int \left(1-\cos^2\theta\right)\operatorname{sen}\theta\,d\theta = \int \left(1-y^2\right)(-dy) \qquad (\text{ para } y = \cos\theta)$$

$$= \int \left(y^2-1\right)dy = \frac{1}{3}y^3 - y + C_1 = \frac{1}{3}\cos^3\theta - \cos\theta + C_1 = \frac{1}{3}\left(\cos^3\theta - 3\cos\theta\right) + C_1 =$$

$$= \frac{1}{3}\left(\cos^2\theta - 3\right)\cos\theta + C_1 = \frac{1}{3}\left[\left(1-x^2\right)-3\right]\sqrt{1-x^2} + C_1 = -\frac{1}{3}\left(x^2+2\right)\sqrt{1-x^2} + C_1.$$

Por lo tanto,

$$\int x^2 \operatorname{arcsen} x\,dx = \frac{x^3}{3}\operatorname{arcsen} x - \frac{1}{3}\int \frac{x^3}{\sqrt{1-x^2}}\,dx = \frac{x^3}{3}\operatorname{arcsen} x - \frac{1}{3}\left[-\frac{1}{3}\left(x^2+2\right)\sqrt{1-x^2}\right] + C =$$

$$= \frac{x^3}{3}\operatorname{arcsen} x + \frac{1}{9}\left(x^2+2\right)\sqrt{1-x^2} + C.$$

□

Ejemplo 2.8.4 *Calcular la integral* $\int x^3 \arctan x\, dx$.

▼ Primero aplicamos la integración por partes

$$\underbrace{\int x^3 \arctan x\, dx}_{\begin{array}{|ccc|} \hline u = \arctan x & \& & dv = x^3\,dx; \\ du = \dfrac{dx}{1+x^2} & \& & v = \dfrac{x^4}{4}. \\ \hline \end{array}} = (\arctan x)\frac{x^4}{4} - \int \frac{x^4}{4}\frac{dx}{1+x^2} = \frac{x^4}{4}\arctan x - \frac{1}{4}\int \frac{x^4}{x^2+1}\,dx.$$

Efectuando la división:

$$\frac{x^4}{x^2+1} = x^2 - 1 + \frac{1}{x^2+1}.$$

Por lo tanto,

$$\int x^3 \arctan x\,dx = \frac{x^4}{4}\arctan x - \frac{1}{4}\int \left(x^2 - 1 + \frac{1}{x^2+1}\right)dx = \frac{x^4}{4}\arctan x - \frac{1}{4}\left[\frac{x^3}{3} - x + \arctan x\right] + C =$$

$$= \frac{1}{4}\left[x^4\arctan x - \frac{x^3}{3} + x - \arctan x\right] + C.$$

□

Ejemplo 2.8.5 *Calcular la integral* $\int x \arccos x \, dx$.

▼ Primero aplicamos la integración por partes

$$\underbrace{\int x \arccos x \, dx}_{\boxed{\begin{array}{lcl} u = \arccos x & \& & dv = x\,dx; \\ du = -\dfrac{dx}{\sqrt{1-x^2}} & \& & v = \dfrac{x^2}{2}. \end{array}}} = (\arccos x)\frac{x^2}{2} - \int -\frac{x^2}{2}\frac{dx}{\sqrt{1-x^2}} = \frac{x^2}{2}\arccos x + \frac{1}{2}\int \frac{x^2}{\sqrt{1-x^2}}\,dx.$$

Ahora, por sustitución trigonométrica:

$$x = \operatorname{sen}\theta \;\Rightarrow\; dx = \cos\theta \, d\theta \;\Rightarrow\; \sqrt{1-x^2} = \sqrt{1-\operatorname{sen}^2\theta} = \cos\theta.$$

Luego,

$$\int \frac{x^2}{\sqrt{1-x^2}}\,dx = \int \frac{\operatorname{sen}^2\theta}{\cos\theta}(\cos\theta)\,d\theta = \int \operatorname{sen}^2\theta\,d\theta = \int \frac{1}{2}(1-\cos 2\theta)\,d\theta = \frac{1}{2}\left[\theta - \frac{1}{2}\operatorname{sen}2\theta\right] + C_1 =$$
$$= \frac{1}{2}\left[\theta - \frac{1}{2}(2\operatorname{sen}\theta\,\cos\theta)\right] + C_1 = \frac{1}{2}\left[\theta - \operatorname{sen}\theta\,\cos\theta\right] + C_1 = \frac{1}{2}\left[\operatorname{arcsen} x - x\sqrt{1-x^2}\right] + C_1.$$

Por lo tanto,

$$\int x \arccos x\,dx = \frac{x^2}{2}\arccos x + \frac{1}{2}\int \frac{x^2}{\sqrt{1-x^2}}\,dx = \frac{x^2}{2}\arccos x + \frac{1}{2}\left(\frac{1}{2}\right)\left(\operatorname{arcsen} x - x\sqrt{1-x^2}\right) + C =$$
$$= \frac{x^2}{2}\arccos x + \frac{1}{4}\operatorname{arcsen} x - \frac{x}{4}\sqrt{1-x^2} + C.$$

□

Ejemplo 2.8.6 *Calcular la integral* $\int x^3 \operatorname{arcsec} x \, dx$.

▼ Aplicamos primero integración por partes.

$$\underbrace{\int x^3 \operatorname{arcsec} x \, dx}_{\boxed{\begin{array}{lcl} u = \operatorname{arcsec} x & \& & dv = x^3\,dx; \\ du = \dfrac{dx}{x\sqrt{x^2-1}} & \& & v = \dfrac{x^4}{4}. \end{array}}} = (\operatorname{arcsec} x)\frac{x^4}{4} - \int \frac{x^4}{4}\frac{dx}{x\sqrt{x^2-1}} = \frac{x^4}{4}\operatorname{arcsec} x - \frac{1}{4}\int \frac{x^3\,dx}{\sqrt{x^2-1}} =$$
$$= \frac{x^4}{4}\operatorname{arcsec} x - \frac{1}{4}\int x^2\frac{x\,dx}{\sqrt{x^2-1}}.$$

De nuevo aplicamos integración por partes:

$$\hat{u} = x^2 \quad \& \quad d\hat{v} = \frac{x\,dx}{\sqrt{x^2-1}} = \left(x^2-1\right)^{-\frac{1}{2}} x\,dx \;\Rightarrow\; d\hat{u} = 2x\,dx \quad \& \quad \hat{v} = \left(x^2-1\right)^{\frac{1}{2}}.$$

Entonces,

$$\int x^3 \operatorname{arcsec} x\,dx = \frac{x^4}{4}\operatorname{arcsec} x - \frac{1}{4}\int x^2\frac{x\,dx}{\sqrt{x^2-1}} = \frac{x^4}{4}\operatorname{arcsec} x - \frac{1}{4}\left[x^2\left(x^2-1\right)^{\frac{1}{2}} - \int \left(x^2-1\right)^{\frac{1}{2}} 2x\,dx\right] =$$
$$= \frac{x^4}{4}\operatorname{arcsec} x - \frac{1}{4}\left[x^2\left(x^2-1\right)^{\frac{1}{2}} - \frac{2}{3}\left(x^2-1\right)^{\frac{3}{2}}\right] + C =$$
$$= \frac{x^4}{4}\operatorname{arcsec} x - \frac{x^2}{4}\left(x^2-1\right)^{\frac{1}{2}} + \frac{1}{6}\left(x^2-1\right)^{\frac{3}{2}} + C =$$
$$= \frac{x^4}{4}\operatorname{arcsec} x - \frac{x^2}{4}\sqrt{x^2-1} + \frac{1}{6}\sqrt{\left(x^2-1\right)^3} + C.$$

□

2.8.4 Cambio de variable y luego fracciones parciales

Trataremos integrales en las que el integrando sea, de preferencia, un cociente de funciones y que presente repetición de alguna función $\phi(x)$; por ejemplo, exponencial, trigonométrica, radical, entre otras.

Al tener una función $\phi(x)$ repetida en el integrando se sugiere aplicar el cambio de variable $u = \phi(x)$.

Ejemplos de estas integrales son:

$$\int \frac{e^x \, \mathrm{d}x}{e^{2x} - 3e^x + 2}; \quad \int \frac{2e^{2x} + e^x + 1}{e^{2x} + 1} \, \mathrm{d}x; \quad \int \frac{\operatorname{sen} x}{\cos x + \cos^2 x} \, \mathrm{d}x.$$

Ejemplo 2.8.7 *Calcular la integral* $\displaystyle\int \frac{e^x \, \mathrm{d}x}{e^{2x} - 3e^x + 2}$.

▼ Como $e^{2x} = (e^x)^2$, entonces en el integrando hay una repetición de la función $\phi(x) = e^x$. Por esto, pensamos en el cambio de variable $u = e^x$. Con $u = e^x \Rightarrow \mathrm{d}u = e^x \, \mathrm{d}x$ y además

$$\int \frac{e^x \, \mathrm{d}x}{e^{2x} - 3e^x + 2} = \int \frac{e^x \, \mathrm{d}x}{(e^x)^2 - 3\,(e^x) + 2} = \int \frac{\mathrm{d}u}{u^2 - 3u + 2}.$$

Ahora aplicamos fracciones parciales.

$$\frac{1}{u^2 - 3u + 2} = \frac{1}{(u-1)(u-2)} = \frac{A}{u-1} + \frac{B}{u-2} = \frac{A(u-2) + B(u-1)}{(u-1)(u-2)}.$$

Igualando polinomios del numerador:

$$1 = A(u-2) + B(u-1). \tag{*}$$

Usando $u = 2$ en (*) $\Rightarrow 1 = A(0) + B(1) \Rightarrow B = 1$.
Usando $u = 1$ en (*) $\Rightarrow 1 = A(-1) + B(0) \Rightarrow -A = 1 \Rightarrow A = -1$.

Luego,

$$\frac{1}{u^2 - 3u + 2} = \frac{A}{u-1} + \frac{B}{u-2} = \frac{-1}{u-1} + \frac{1}{u-2} = \frac{1}{u-2} - \frac{1}{u-1}.$$

Entonces,

$$\int \frac{1}{u^2 - 3u + 2} = \int \left(\frac{1}{u-2} - \frac{1}{u-1} \right) \mathrm{d}u = \int \frac{\mathrm{d}u}{u-2} - \int \frac{\mathrm{d}u}{u-1} = \ln(u-2) - \ln(u-1) + C =$$
$$= \ln\left(\frac{u-2}{u-1} \right) + C.$$

Por lo tanto, como $u = e^x$,

$$\int \frac{e^x \, \mathrm{d}x}{e^{2x} - 3e^x + 2} = \ln\left(\frac{e^x - 2}{e^x - 1} \right) + C.$$

□

Ejemplo 2.8.8 *Calcular la integral* $\displaystyle\int \frac{2e^{2x} + e^x + 1}{e^{2x} + 1} \, \mathrm{d}x$.

▼ Como $e^{2x} = (e^x)^2$, entonces en el integrando hay una repetición de la función exponencial $\phi(x) = e^x$. Esto nos sugiere el cambio de variable $u = e^x$.

$$u = e^x \Rightarrow x = \ln u \quad \& \quad \mathrm{d}x = \frac{\mathrm{d}u}{u}.$$

Luego,

$$\int \frac{2e^{2x} + e^x + 1}{e^{2x} + 1}\,\mathrm{d}x = \int \frac{2\,(e^x)^2 + (e^x) + 1}{(e^x)^2 + 1}(\mathrm{d}x) = \int \frac{2u^2 + u + 1}{u^2 + 1}\left(\frac{\mathrm{d}u}{u}\right) = \int \frac{2u^2 + u + 1}{u\,(u^2 + 1)}\,\mathrm{d}u.$$

Ahora aplicamos fracciones parciales.

$$\frac{2u^2 + u + 1}{u\,(u^2 + 1)}\,\mathrm{d}u = \frac{A}{u} + \frac{Bu + C}{u^2 + 1} = \frac{A\left(u^2 + 1\right) + (Bu + C)u}{u\,(u^2 + 1)}.$$

Igualando los polinomios de los numeradores:

$$2u^2 + u + 1 = (A + B)u^2 + (C)u + (A).$$

Igualamos coeficientes de términos semejantes:

$$A + B = 2; \quad C = 1; \quad A = 1.$$

Entonces:

$$B = 1; \quad C = 1; \quad A = 1$$

Luego,

$$\frac{2u^2 + u + 1}{u\,(u^2 + 1)} = \frac{A}{u} + \frac{Bu + C}{u^2 + 1} = \frac{1}{u} + \frac{u + 1}{u^2 + 1}.$$

Por lo tanto, como $u = e^x$,

$$\int \frac{2e^{2x} + e^x + 1}{e^{2x} + 1}\,\mathrm{d}x = \int \left(\frac{1}{u} + \frac{u + 1}{u^2 + 1}\right)\mathrm{d}u = \int \left(\frac{1}{u} + \frac{u}{u^2 + 1} + \frac{1}{u^2 + 1}\right)\mathrm{d}u =$$
$$= \ln u + \frac{1}{2}\ln\left(u^2 + 1\right) + \arctan u + C = \ln e^x + \ln\left(e^{2x} + 1\right)^{\frac{1}{2}} + \arctan\left(e^x\right) + C =$$
$$= x + \ln\sqrt{e^{2x} + 1} + \arctan\left(e^x\right) + C.$$

□

Ejemplo 2.8.9 *Calcular la integral* $\displaystyle\int \frac{\operatorname{sen} x}{\cos x + \cos^2 x}\,\mathrm{d}x$.

▼ Tenemos:

$$\frac{\mathrm{d}}{\mathrm{d}x}(\cos x) = -\operatorname{sen} x \Rightarrow (\operatorname{sen} x)\,\mathrm{d}x = -\,\mathrm{d}(\cos x).$$

Observe una repetición de la función $\cos x$ en el integrando. Por esto aplicamos el cambio de variable $y = \cos x$.

$$\int \frac{\operatorname{sen} x}{\cos x + \cos^2 x}\,\mathrm{d}x = \int \frac{-\,\mathrm{d}(\cos x)}{(\cos x) + (\cos x)^2} = \int \frac{-\,\mathrm{d}y}{y + y^2} = -\int \frac{\mathrm{d}y}{y + y^2}.$$

Ahora aplicamos fracciones parciales:

$$\frac{1}{y + y^2} = \frac{1}{y(1 + y)} = \frac{A}{y} + \frac{B}{1 + y} = \frac{A(1 + y) + By}{y(1 + y)} = \frac{(A + B)y + (A)}{y(1 + y)}.$$
$$1 = (A + B)y + (A) \Leftrightarrow A + B = 0 \quad \& \quad A = 1 \Leftrightarrow A = 1 \quad \& \quad B = -1.$$

Luego,

$$\frac{1}{y+y^2}=\frac{A}{y}+\frac{B}{1+y}=\frac{1}{y}-\frac{1}{1+y}.$$

Por lo que,

$$-\int\frac{dy}{y+y^2}=-\int\left(\frac{1}{y}-\frac{1}{1+y}\right)dy=\int\left(\frac{1}{1+y}-\frac{1}{y}\right)dy=\ln(1+y)-\ln y+C=\ln\left(\frac{1+y}{y}\right)+C.$$

Por lo tanto,

$$\int\frac{\operatorname{sen}x}{\cos x+\cos^2x}\,dx=-\int\frac{dy}{y+y^2}=\ln\left(\frac{1+y}{y}\right)+C=\ln\left(\frac{1}{y}+1\right)+C=$$
$$=\ln\left(\frac{1}{\cos x}+1\right)+C=\ln(\sec x+1)+C.$$

□

Ejemplo 2.8.10 *Calcular la integral* $\displaystyle\int\frac{\sqrt{x+1}+2}{(x+1)^2-\sqrt{x+1}}\,dx$.

▼ Es notoria la repetición del radical $\sqrt{x+1}$ o bien de $(x+1)$. Optamos por eliminar el radical, por lo que aplicaremos el cambio de variable $\sqrt{x+1}=y$. Veamos:

$$\sqrt{x+1}=y\ \Rightarrow\ x+1=y^2\ \Rightarrow\ x=y^2-1\ \&\ dx=2y\,dy.$$

Luego,

$$\int\frac{\sqrt{x+1}+2}{(x+1)^2-\sqrt{x+1}}\,dx=\int\frac{y+2}{(y^2)^2-y}(2y\,dy)=\int\frac{2y(y+2)}{y^4-y}\,dy=\int\frac{2y(y+2)}{y\,(y^3-1)}\,dy=\int\frac{2(y+2)}{y^3-1}\,dy.$$

Aplicamos fracciones parciales a continuación

$$\frac{2(y+2)}{y^3-1}=\frac{2y+4}{(y-1)\,(y^2+y+1)}=\frac{A}{y-1}+\frac{By+C}{y^2+y+1}=\frac{A\left(y^2+y+1\right)+(By+C)(y-1)}{(y-1)\,(y^2+y+1)};$$
$$2y+4=A\left(y^2+y+1\right)+B\left(y^2-y\right)+C(y-1);$$
$$2y+4=(A+B)y^2+(A-B+C)y+(A-C);$$
$$A+B=0;\quad A-B+C=2;\quad A-C=4;$$
$$B=-A;\quad A-B+C=2;\quad C=A-4;$$
$$A-(-A)+(A-4)=2;$$
$$A+A+A=2+4;$$
$$3A=6\ \Rightarrow\ A=2;$$
$$A=2\ \Rightarrow\ B=-A=-2\quad\&\quad C=A-4=2-4=-2.$$

Luego,

$$\frac{2(y+2)}{y^3-1}=\frac{A}{y-1}+\frac{By+C}{y^2+y+1}=\frac{2}{y-1}+\frac{-2y-2}{y^2+y+1}=\frac{2}{y-1}-\frac{2y+2}{y^2+y+1}.$$

Por lo tanto,

$$\int\frac{\sqrt{x+1}+2}{(x+1)^2-\sqrt{x+1}}\,dx=\int\frac{2(y+2)}{y^3-1}\,dy=\int\left(\frac{2}{y-1}-\frac{2y+2}{y^2+y+1}\right)dy=\int\left(\frac{2}{y-1}-\frac{2y+1+1}{y^2+y+1}\right)dy=$$
$$=\int\left(\frac{2}{y-1}-\frac{2y+1}{y^2+y+1}-\frac{1}{y^2+y+1}\right)dy=$$

$$
\begin{aligned}
&= 2\ln(y-1) - \ln\left(y^2+y+1\right) - \int \frac{\mathrm{d}y}{\left(y+\frac{1}{2}\right)^2 + \frac{3}{4}} = \\
&= \ln(y-1)^2 - \ln\left(y^2+y+1\right) - \int \frac{\mathrm{d}y}{\left(y+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \\
&= \ln\left[\frac{(y-1)^2}{y^2+y+1}\right] - \frac{2}{\sqrt{3}}\arctan\frac{2}{\sqrt{3}}\left(y+\frac{1}{2}\right) + K = \\
&= \ln\left[\frac{(y-1)^2}{y^2+y+1}\right] - \frac{2}{\sqrt{3}}\arctan\left(\frac{2y+1}{\sqrt{3}}\right) + K = \\
&= \ln\left[\frac{(\sqrt{x+1}-1)^2}{(x+2)+\sqrt{x+1}}\right] - \frac{2}{3}\arctan\left(\frac{2\sqrt{x+1}+1}{3}\right) + K.
\end{aligned}
$$

□

Ejemplo 2.8.11 *Calcular la integral* $\int \sqrt{\tan x}\,\mathrm{d}x$.

▼ Aquí no hay una función repetida; aún más, sólo tenemos una función $\sqrt{\tan x}$, la cual tomamos para un cambio de variable.

$$
\sqrt{\tan x} = y \;\Rightarrow\; \tan x = y^2 \;\Rightarrow\; x = \arctan y^2 \;\Rightarrow\; \mathrm{d}x = \frac{2y\,\mathrm{d}y}{1+y^4}.
$$

Entonces,

$$
\int \sqrt{\tan x}\,\mathrm{d}x = \int y\left(\frac{2y\,\mathrm{d}y}{1+y^4}\right) = \int \frac{2y^2}{y^4+1}\,\mathrm{d}y.
$$

Ahora aplicamos fracciones parciales, para lo cual factorizamos el polinomio denominador

$$
\begin{aligned}
y^4+1 = y^4+2y^2+1-2y^2 = \left(y^2+1\right)^2 - \left(\sqrt{2}y\right)^2 = \left(y^2+1+\sqrt{2}y\right)\left(y^2+1-\sqrt{2}y\right) = \\
= \left(y^2+\sqrt{2}y+1\right)\left(y^2-\sqrt{2}y+1\right).
\end{aligned}
$$

Luego,

$$
\begin{aligned}
\frac{2y^2}{y^4+1} &= \frac{2y^2}{\left(y^2+\sqrt{2}y+1\right)\left(y^2-\sqrt{2}y+1\right)} = \frac{Ay+B}{y^2+\sqrt{2}y+1} + \frac{Cy+D}{y^2-\sqrt{2}y+1} = \\
&= \frac{(Ay+B)\left(y^2-\sqrt{2}y+1\right) + (Cy+D)\left(y^2+\sqrt{2}y+1\right)}{\left(y^2+\sqrt{2}y+1\right)\left(y^2-\sqrt{2}y+1\right)}.
\end{aligned}
$$

Igualdad que se cumple cuando

$$
\begin{aligned}
2y^2 &= A\left(y^3-\sqrt{2}y^2+y\right) + B\left(y^2-\sqrt{2}y+1\right) + C\left(y^3+\sqrt{2}y^2+y\right) + D\left(y^2+\sqrt{2}y+1\right); \\
2y^2 &= (A+C)y^3 + \left(-\sqrt{2}A+B+\sqrt{2}C+D\right)y^2 + \left(A-\sqrt{2}B+C+\sqrt{2}D\right)y + (B+D).
\end{aligned}
$$

Igualdad de polinomios que ocurre cuando

$$
\begin{cases}
A+C=0; & \text{(Ec. 1)}\\
-\sqrt{2}A+B+\sqrt{2}C+D=2; & \text{(Ec. 2)}\\
A-\sqrt{2}B+C+\sqrt{2}D=0; & \text{(Ec. 3)}\\
B+D=0. & \text{(Ec. 4)}
\end{cases}
$$

Aplicando (Ec. 4) en (Ec. 2) se obtiene

$$-\sqrt{2}A + (B + D) + \sqrt{2}C = 2 \Leftrightarrow -\sqrt{2}A + \sqrt{2}C = 2 \Leftrightarrow -A + C = \sqrt{2}.$$

Aplicando (Ec. 1) en (Ec. 3) se obtiene

$$(A + C) - \sqrt{2}B + \sqrt{2}D = 0 \Leftrightarrow -\sqrt{2}B + \sqrt{2}D = 0 \Leftrightarrow -B + D = 0.$$

Tenemos el nuevo sistema de ecuaciones

$$\begin{cases} A + C = 0; & \text{(Ec. 1)} \\ -A + C = \sqrt{2}; & \text{(Ec. 2')} \\ -B + D = 0; & \text{(Ec. 3')} \\ B + D = 0. & \text{(Ec. 4)} \end{cases}$$

Ahora bien,

$$\begin{aligned} &\text{(Ec. 1)} + \text{(Ec. 2')} \Rightarrow 2C = \sqrt{2} && \Rightarrow C = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}} \quad \& \quad A = -C = -\frac{1}{\sqrt{2}}; \\ &\text{(Ec. 3')} + \text{(Ec. 4')} \Rightarrow 2D = 0 && \Rightarrow D = 0 \quad \& \quad B = -D = 0. \end{aligned}$$

Entonces, la solución del sistema de ecuaciones es

$$A = -\frac{1}{\sqrt{2}}, \quad B = 0; \quad C = \frac{1}{\sqrt{2}} \,\&\, D = 0.$$

Luego,

$$\frac{2y^2}{y^4+1} = \frac{-\frac{1}{\sqrt{2}}y + 0}{y^2 + \sqrt{2}y + 1} + \frac{\frac{1}{\sqrt{2}}y + 0}{y^2 - \sqrt{2}y + 1} = \frac{-\frac{1}{\sqrt{2}}y}{y^2 + \sqrt{2}y + 1} + \frac{\frac{1}{\sqrt{2}}y + 0}{y^2 - \sqrt{2}y + 1}.$$

Por lo que,

$$\begin{aligned}
&\int \frac{2y^2}{y^4+1}\,\mathrm{d}y = -\frac{1}{\sqrt{2}}\int \frac{y\,\mathrm{d}y}{y^2+\sqrt{2}y+1} + \frac{1}{\sqrt{2}}\int \frac{y\,\mathrm{d}y}{y^2-\sqrt{2}y+1} = \\
&= -\frac{1}{2\sqrt{2}}\int \frac{2y}{y^2+\sqrt{2}y+1}\,\mathrm{d}y + \frac{1}{2}\sqrt{2}\int \frac{2y}{y^2-\sqrt{2}y+1}\,\mathrm{d}y = \\
&= -\frac{1}{2\sqrt{2}}\int \frac{\left(2y+\sqrt{2}\right)-\sqrt{2}}{y^2+\sqrt{2}y+1}\,\mathrm{d}y + \frac{1}{2\sqrt{2}}\int \frac{\left(2y-\sqrt{2}\right)-\sqrt{2}}{y^2-\sqrt{2}y+1}\,\mathrm{d}y = \\
&= -\frac{1}{2\sqrt{2}}\int \left(\frac{2y+\sqrt{2}}{y^2+\sqrt{2}y+1} - \frac{\sqrt{2}}{y^2+\sqrt{2}y+1}\right)\mathrm{d}y + \frac{1}{2\sqrt{2}}\int \left(\frac{2y-\sqrt{2}}{y^2-\sqrt{2}y+1} + \frac{\sqrt{2}}{y^2-\sqrt{2}y+1}\right)\mathrm{d}y = \\
&= -\frac{1}{2\sqrt{2}}\ln\left(y^2+\sqrt{2}y+1\right) + \frac{1}{2}\int \frac{\mathrm{d}y}{y^2+\sqrt{2}y+1} + \frac{1}{2\sqrt{2}}\ln\left(y^2-\sqrt{2}y+1\right) + \frac{1}{2}\int \frac{\mathrm{d}y}{y^2-\sqrt{2}y+1} = \\
&= \frac{1}{2\sqrt{2}}\left[\ln\left(y^2-\sqrt{2}y+1\right) - \ln\left(y^2+\sqrt{2}y+1\right)\right] + \frac{1}{2}\left(\int \frac{\mathrm{d}y}{y^2+\sqrt{2}y+1} + \int \frac{\mathrm{d}y}{y^2-\sqrt{2}y+1}\right) = \\
&= \frac{1}{2\sqrt{2}}\ln\left(\frac{y^2-\sqrt{2}y+1}{y^2+\sqrt{2}y+1}\right) + \frac{1}{2}\left[\int \frac{\mathrm{d}y}{\left(y+\frac{1}{\sqrt{2}}\right)^2+\frac{1}{2}} + \int \frac{\mathrm{d}y}{\left(y-\frac{1}{\sqrt{2}}\right)^2+\frac{1}{2}}\right] = \\
&= \frac{1}{2\sqrt{2}}\ln\left(\frac{y^2-\sqrt{2}y+1}{y^2+\sqrt{2}y+1}\right) + \frac{1}{2}\left[\int \frac{\mathrm{d}y}{\left(y+\frac{1}{\sqrt{2}}\right)^2+\left(\frac{1}{\sqrt{2}}\right)^2} + \int \frac{\mathrm{d}y}{\left(y-\frac{1}{\sqrt{2}}\right)^2+\left(\frac{1}{\sqrt{2}}\right)^2}\right] =
\end{aligned}$$

$$= \frac{1}{2\sqrt{2}} \ln\left(\frac{y^2 - \sqrt{2}y + 1}{y^2 + \sqrt{2}y + 1}\right) + \frac{1}{2}\left[\sqrt{2}\arctan\sqrt{2}\left(y + \frac{1}{\sqrt{2}}\right) + \sqrt{2}\arctan\sqrt{2}\left(y - \frac{1}{\sqrt{2}}\right)\right] + K =$$

$$= \frac{1}{2\sqrt{2}} \ln\left(\frac{y^2 - \sqrt{2}y + 1}{y^2 + \sqrt{2}y + 1}\right) + \frac{\sqrt{2}}{2}\left[\arctan\left(\sqrt{2}y + 1\right) + \arctan\left(\sqrt{2}y - 1\right)\right] + K =$$

$$= \frac{1}{\sqrt{2}} \ln\left(\frac{y^2 - \sqrt{2}y + 1}{y^2 + \sqrt{2}y + 1}\right)^{\frac{1}{2}} + \frac{1}{\sqrt{2}}\left[\arctan\left(\sqrt{2}y + 1\right) + \arctan\left(\sqrt{2}y - 1\right)\right] + K =$$

$$= \frac{1}{\sqrt{2}}\left[\ln\left(\frac{y^2 - \sqrt{2}y + 1}{y^2 + \sqrt{2}y + 1}\right]^{\frac{1}{2}} + \arctan\left(\sqrt{2}y + 1\right) + \arctan\left(\sqrt{2}y - 1\right)\right] + K.$$

Debido a que $y = \sqrt{\tan x}$, concluimos:

$$\int \sqrt{\tan x}\, dx = \frac{1}{\sqrt{2}}\left[\ln\left(\frac{\tan x - \sqrt{2\tan x} + 1}{\tan x + \sqrt{2\tan x} + 1}\right)^{\frac{1}{2}} + \arctan\left(\sqrt{2\tan x} + 1\right) + \arctan\left(\sqrt{2\tan x} - 1\right)\right] + K.$$

□

2.8.5 Cambio de variable especial y luego la integral de una función racional

A partir de una función racional de senos y cosenos, vamos a presentar un cambio de variable que nos permite convertir nuestra integral en una integral de una función racional.

Primero obtendremos un par de identidades en las que se relacionan las funciones seno, coseno y tangente, para luego ejemplificar el uso de dichas identidades para calcular este tipo de integrales.

- Debido a que

$$\operatorname{sen} 2\alpha = 2\operatorname{sen}\alpha\ \cos\alpha \quad \& \quad \cos 2\alpha = \cos^2\alpha - \operatorname{sen}^2\alpha;$$

$$\tan 2\alpha = \frac{\operatorname{sen} 2\alpha}{\cos 2\alpha} = \frac{2\operatorname{sen}\alpha\ \cos\alpha}{\cos^2\alpha - \operatorname{sen}^2\alpha} = \frac{\frac{2\operatorname{sen}\alpha\ \cos\alpha}{\cos^2\alpha}}{\frac{\cos^2\alpha}{\cos^2\alpha} - \frac{\operatorname{sen}^2\alpha}{\cos^2\alpha}} = \frac{2\tan\alpha}{1 - \tan^2\alpha}.$$

Y considerando $2\alpha = \theta$, es decir $\alpha = \dfrac{\theta}{2}$,

$$\tan 2\alpha = \frac{2\tan\alpha}{1 - \tan^2\alpha} \Rightarrow \tan\theta = \frac{2\tan\frac{\theta}{2}}{1 - \tan^2\frac{\theta}{2}}.$$

Ahora, si $m = \tan\frac{\theta}{2}$ y $\tan\theta = \dfrac{\operatorname{sen}\theta}{\cos\theta}$:

$$\frac{\operatorname{sen}\theta}{\cos\theta} = \frac{2m}{1 - m^2} \Rightarrow \frac{\operatorname{sen}^2\theta}{\cos^2\theta} = \left(\frac{2m}{1 - m^2}\right)^2 = \frac{4m^2}{(1 - m^2)^2} \Rightarrow$$

$$\Rightarrow \left(1 - m^2\right)^2 \operatorname{sen}^2\theta = 4m^2\cos^2\theta \Rightarrow \left(1 - m^2\right)^2 \operatorname{sen}^2\theta = 4m^2\left(1 - \operatorname{sen}^2\theta\right) = 4m^2 - 4m^2\operatorname{sen}^2\theta \Rightarrow$$

$$\Rightarrow \left(1 - m^2\right)^2 \operatorname{sen}^2\theta + 4m^2\operatorname{sen}^2\theta = 4m^2 \Rightarrow \left[\left(1 - m^2\right)^2 + 4m^2\right]\operatorname{sen}^2\theta = 4m^2 \Rightarrow$$

$$\Rightarrow \left[1 - 2m^2 + m^4 + 4m^2\right]\operatorname{sen}^2\theta = 4m^2 \Rightarrow \left[1 + 2m^2 + m^4\right]\operatorname{sen}^2\theta = 4m^2 \Rightarrow$$

$$\Rightarrow \left(1 + m^2\right)^2 \operatorname{sen}^2\theta = (2m)^2 \Rightarrow \left(1 + m^2\right)\operatorname{sen}\theta = 2m \Rightarrow$$

$$\Rightarrow \operatorname{sen}\theta = \frac{2m}{1 + m^2}.$$

Además,

$$\operatorname{sen}^2\theta + \cos^2\theta = 1 \Rightarrow \cos^2\theta = 1 - \operatorname{sen}^2\theta = 1 - \left(\frac{2m}{1+m^2}\right)^2 = 1 - \frac{4m^2}{(1+m^2)^2} \Rightarrow$$
$$\Rightarrow \cos^2\theta = \frac{\left(1+m^2\right)^2 - 4m^2}{(1+m^2)^2} = \frac{1+2m^2+m^4-4m^2}{(1+m^2)^2} = \frac{1-2m^2+m^4}{(1+m^2)^2} = \frac{\left(1-m^2\right)^2}{(1+m^2)^2} \Rightarrow$$
$$\Rightarrow \cos\theta = \frac{1-m^2}{1+m^2}.$$

donde $m = \tan\dfrac{\theta}{2}$. Esto es,

$$\operatorname{sen}\theta = \frac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}} \qquad \& \qquad \cos\theta = \frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}.$$

- Ahora, ejemplifiquemos el uso de estas identidades para calcular integrales.

Ejemplo 2.8.12 *Calcular la integral* $\displaystyle\int \frac{dx}{\operatorname{sen}x + \cos x}$.

▼

$$\operatorname{sen}x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} \quad \& \quad \cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}} \Rightarrow$$
$$\Rightarrow \operatorname{sen}x + \cos x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} + \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}} = \frac{2\tan\frac{x}{2} + 1 - \tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}} = -\frac{\tan^2\frac{x}{2} - 2\tan\frac{x}{2} - 1}{\tan^2\frac{x}{2}+1} \Rightarrow$$
$$\Rightarrow \frac{1}{\operatorname{sen}x+\cos x} = -\frac{\tan^2\frac{x}{2}+1}{\tan^2\frac{x}{2} - 2\tan\frac{x}{2} - 1} \Rightarrow$$
$$\Rightarrow \int \frac{dx}{\operatorname{sen}x+\cos x} = -\int \frac{\tan^2\frac{x}{2}+1}{\tan^2\frac{x}{2} - 2\tan\frac{x}{2} - 1}\,dx.$$

Ahora, utilizando el cambio de variable $u = \tan\dfrac{x}{2}$:

$$u = \tan\frac{x}{2} \Rightarrow \frac{x}{2} = \arctan u \Rightarrow x = 2\arctan u \Rightarrow dx = \frac{2\,du}{1+u^2}.$$

Luego,

$$\int \frac{dx}{\operatorname{sen}x+\cos x} = -\int \frac{u^2+1}{u^2-2u-1}\,\frac{2\,du}{1+u^2} = -2\int \frac{du}{u^2-2u-1}.$$

Aplicamos fracciones parciales.

$$u^2-2u-1 = \left(u^2-2u+1\right) - 2 = (u-1)^2 - \left(\sqrt{2}\right)^2 = \left(u-1-\sqrt{2}\right)\left(u-1+\sqrt{2}\right) \Rightarrow$$
$$\Rightarrow \frac{1}{u^2-2u-1} = \frac{1}{\left(u-1-\sqrt{2}\right)\left(u-1+\sqrt{2}\right)} =$$
$$= \frac{A}{u-1-\sqrt{2}} + \frac{B}{u-1+\sqrt{2}} = \frac{A\left(u-1+\sqrt{2}\right) + B\left(u-1-\sqrt{2}\right)}{\left(u-1-\sqrt{2}\right)\left(u-1+\sqrt{2}\right)} \Rightarrow$$
$$\Rightarrow 1 = A\left(u-1+\sqrt{2}\right) + B\left(u-1-\sqrt{2}\right) = (A+B)u + \left(-1+\sqrt{2}\right)A + \left(-1-\sqrt{2}\right)B \Rightarrow$$

$$\Rightarrow A+B=0 \quad \& \quad \left(-1+\sqrt{2}\right)A-\left(1+\sqrt{2}\right)B=1 \;\Rightarrow$$
$$\Rightarrow B=-A \quad \& \quad \left(-1+\sqrt{2}\right)A-\left(1+\sqrt{2}\right)B=1 \;\Rightarrow$$
$$\Rightarrow \left(-1+\sqrt{2}\right)A-\left(1+\sqrt{2}\right)(-A)=1 \;\Rightarrow$$
$$\Rightarrow \left[\left(-1+\sqrt{2}\right)+\left(1+\sqrt{2}\right)\right]A=1 \;\Rightarrow\; 2\sqrt{2}A=1 \;\Rightarrow$$
$$\Rightarrow A=\frac{1}{2\sqrt{2}} \quad \& \quad B=-A \;\Rightarrow\; B=\frac{-1}{2\sqrt{2}}.$$

Luego,

$$\frac{1}{u^2-2u-1}=\frac{A}{u-1-\sqrt{2}}+\frac{B}{u-1+\sqrt{2}}=\frac{\frac{1}{2\sqrt{2}}}{u-1-\sqrt{2}}+\frac{\frac{-1}{2\sqrt{2}}}{u-1+\sqrt{2}}=\frac{1}{2\sqrt{2}}\left[\frac{1}{u-1-\sqrt{2}}-\frac{1}{u-1+\sqrt{2}}\right].$$

Por lo tanto,

$$\int\frac{\mathrm{d}x}{\operatorname{sen}x+\cos x}=-2\int\frac{\mathrm{d}u}{u^2-2u-1}=-2\int\frac{1}{2\sqrt{2}}\left[\frac{1}{u-1-\sqrt{2}}-\frac{1}{u-1+\sqrt{2}}\right]\mathrm{d}u=$$
$$=-\frac{2}{2\sqrt{2}}\left[\ln\left(u-1-\sqrt{2}\right)-\ln\left(u-1+\sqrt{2}\right)\right]+K=$$
$$=-\frac{1}{\sqrt{2}}\left[\ln\left(u-1-\sqrt{2}\right)-\ln\left(u-1+\sqrt{2}\right)\right]+K=$$
$$=\frac{1}{\sqrt{2}}\left[\ln\left(u-1+\sqrt{2}\right)-\ln\left(u-1-\sqrt{2}\right)\right]+K=\frac{1}{\sqrt{2}}\ln\left[\frac{u-1+\sqrt{2}}{u-1-\sqrt{2}}\right]+K=$$
$$=\frac{1}{\sqrt{2}}\ln\left[\frac{\tan\frac{x}{2}-1+\sqrt{2}}{\tan\frac{x}{2}-1-\sqrt{2}}\right]+K.$$

Esto es,

$$\int\frac{\mathrm{d}x}{\operatorname{sen}x+\cos x}=\frac{1}{\sqrt{2}}\ln\left[\frac{\tan\frac{x}{2}-1+\sqrt{2}}{\tan\frac{x}{2}-1-\sqrt{2}}\right]+K.$$

□

Ejemplo 2.8.13 *Calcular la integral* $\displaystyle\int\frac{\mathrm{d}x}{\tan x-1}$.

▼ Por ser $\tan x=\dfrac{\operatorname{sen}x}{\cos x}$:

$$\int\frac{\mathrm{d}x}{\tan x-1}=\int\frac{\mathrm{d}x}{\frac{\operatorname{sen}x}{\cos x}-1}=\int\frac{\cos x}{\operatorname{sen}x-\cos x}\,\mathrm{d}x.$$

Aplicando las identidades

$$\operatorname{sen}x=\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} \quad \& \quad \cos x=\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}},$$

se obtiene:

$$\int\frac{\mathrm{d}x}{\tan x-1}=\int\frac{\cos x}{\operatorname{sen}x-\cos x}\,\mathrm{d}x=\int\frac{\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}}{\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}-\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}}\,\mathrm{d}x=\int\frac{1-\tan^2\frac{x}{2}}{2\tan\frac{x}{2}-\left(1-\tan^2\frac{x}{2}\right)}\,\mathrm{d}x=$$
$$=\int\frac{-\tan^2\frac{x}{2}+1}{\tan^2\frac{x}{2}+2\tan\frac{x}{2}-1}\,\mathrm{d}x.$$

Si ahora aplicamos un cambio de variable:

$$u=\tan\tfrac{x}{2}\ \Rightarrow\ \tfrac{x}{2}=\arctan u\ \Rightarrow\ x=2\arctan u\ \Rightarrow\ \mathrm{d}x=\frac{2\,\mathrm{d}u}{1+u^2},$$

con lo cual:

$$\int\frac{-\tan^2\frac{x}{2}+1}{\tan^2\frac{x}{2}+2\tan\frac{x}{2}-1}\,\mathrm{d}x=\int\frac{-u^2+1}{u^2+2u-1}\left(\frac{2\,\mathrm{d}u}{1+u^2}\right)=\int\frac{-2u^2+2}{(u^2+2u-1)\,(u^2+1)}\,\mathrm{d}u.$$

Ahora, fracciones parciales:

$$\frac{-2u^2+2}{(u^2+2u-1)\,(u^2+1)}=\frac{Au+B}{u^2+2u-1}+\frac{Cu+D}{u^2+1}\ \Rightarrow$$
$$\Rightarrow\ -2u^2+2=A\left(u^3+u\right)+B\left(u^2+1\right)+C\left(u^3+2u^2-u\right)+D\left(u^2+2u-1\right)=$$
$$=(A+C)u^3+(B+2C+D)u^2+(A-C+2D)u+(B-D).$$

Esta igualdad de polinomios sucede cuando:

$$A+C=0;\quad B+2C+D=-2;\quad A-C+2D=0\quad \&\quad B-D=2.$$

Su solución es

$$A=1,\quad B=1,\quad C=-1\quad \&\quad D=-1.$$

Luego,

$$\frac{-2u^2+2}{(u^2+2u-1)\,(u^2+1)}=\frac{Au+B}{u^2+2u-1}+\frac{Cu+D}{u^2+1}=$$
$$=\frac{u+1}{u^2+2u-1}+\frac{-u-1}{u^2+1}=\frac{u+1}{u^2+2u-1}-\frac{u+1}{u^2+1}.$$

Entonces,

$$\int\frac{-2u^2+2}{(u^2+2u-1)\,(u^2+1)}\,\mathrm{d}u=\int\left[\frac{u+1}{u^2+2u-1}-\frac{u+1}{u^2+1}\right]\mathrm{d}u=$$
$$=\frac{1}{2}\int\frac{(2u+2)\,\mathrm{d}u}{u^2+2u-1}-\frac{1}{2}\int\frac{2u\,\mathrm{d}u}{u^2+1}-\int\frac{\mathrm{d}u}{u^2+1}=$$
$$=\frac{1}{2}\ln\left(u^2+2u-1\right)-\frac{1}{2}\ln\left(u^2+1\right)-\arctan u+K=$$
$$=\frac{1}{2}\ln\left[\frac{u^2+2u-1}{u^2+1}\right]-\arctan u+K.$$

Por lo tanto,

$$\int\frac{\mathrm{d}x}{\tan x-1}=\int\frac{-\tan^2\frac{x}{2}+1}{\tan^2\frac{x}{2}+2\tan\frac{x}{2}-1}\,\mathrm{d}x=\int\frac{-2u^2+2}{(u^2+2u-1)\,(u^2+1)}\,\mathrm{d}u=$$
$$=\frac{1}{2}\ln\left[\frac{u^2+2u-1}{u^2+1}\right]-\arctan u+K=$$
$$=\frac{1}{2}\ln\left[\frac{\tan^2\frac{x}{2}+2\tan\frac{x}{2}-1}{\tan^2\frac{x}{2}+1}\right]-\arctan\left(\tan\tfrac{x}{2}\right)+K=$$
$$=\ln\left[\frac{\tan^2\frac{x}{2}+2\tan\frac{x}{2}-1}{\sec^2\frac{x}{2}}\right]^{\frac{1}{2}}-\left(\tfrac{x}{2}\right)+K=$$
$$=\ln\sqrt{\left(\tan^2\frac{x}{2}+2\tan\frac{x}{2}-1\right)\cos^2\frac{x}{2}}-\left(\frac{x}{2}\right)+K=$$

$$= \ln \sqrt{\operatorname{sen}^2 \frac{x}{2} + 2 \operatorname{sen} \frac{x}{2} \ \cos \frac{x}{2} - \cos^2 \frac{x}{2}} - \left(\frac{x}{2}\right) + K =$$
$$= \ln \sqrt{2 \operatorname{sen} \frac{x}{2} \ \cos \frac{x}{2} - \left(\cos^2 \frac{x}{2} - \operatorname{sen}^2 \frac{x}{2}\right)} - \left(\frac{x}{2}\right) + K =$$
$$= \ln \sqrt{\operatorname{sen} 2\left(\frac{x}{2}\right) - \cos 2\left(\frac{x}{2}\right)} - \left(\frac{x}{2}\right) + K =$$
$$= \ln \sqrt{\operatorname{sen} x - \cos x} - \frac{x}{2} + K.$$

□

Ejercicios 2.8.1 *Miscelanea. Soluciones en la página* 17

1. $\int e^{\sqrt{x}}\, dx.$
2. $\int \cos \sqrt{x-1}\, dx.$
3. $\int \operatorname{sen} \sqrt[3]{x}\, dx.$
4. $\int x \operatorname{arcsen} x\, dx.$
5. $\int x^2 \arctan x\, dx.$
6. $\int e^{\sqrt[3]{x+2}}\, dx.$
7. $\int x^2 \operatorname{arcsec} x\, dx.$
8. $\int \arctan \sqrt{x}\, dx.$
9. $\int \frac{x^{\frac{2}{3}} - 2x^{\frac{1}{3}} + 1}{x+1}\, dx.$
10. $\int \frac{2x + 3\sqrt{x+1}}{2x - 3\sqrt{x+1}}\, dx.$
11. $\int \frac{3\cos x}{\operatorname{sen}^2 x + \operatorname{sen} x - 2}\, dx.$
12. $\int \frac{e^x}{e^{2x} - 3e^x + 2}.$
13. $\int \frac{\sec^2 x}{\tan^2 x + \tan x}\, dx.$
14. $\int \frac{\operatorname{sen} 2x + 2 \operatorname{sen} x}{\cos^3 x + \cos x}\, dx.$
15. $\int \frac{2e^{2x}}{(e^x - 1)(e^{2x} + 1)}\, dx.$
16. $\int x \arctan\left(\frac{1}{x+1}\right) dx.$
17. $\int x \arctan\left(\frac{1}{x^2+1}\right) dx.$
18. $\int \frac{4e^{2x}}{e^{4x} - 1}\, dx.$
19. $\int \frac{\cos x}{5 + 4\cos x}\, dx.$
20. $\int \frac{dx}{\operatorname{sen} x + \cos x + 1}.$
21. $\int \frac{1 - \operatorname{sen} x}{1 + \operatorname{sen} x}\, dx.$
22. $\int \frac{2\cos x - 5 \operatorname{sen} x}{3\cos x + 4 \operatorname{sen} x}\, dx.$

Ejercicios 2.8.1 *Miscelanea.* *Preguntas, página 16*

1. $2\left(\sqrt{x}-1\right)e^{\sqrt{x}}+C.$
2. $2\sqrt{x-1}\,\operatorname{sen}\sqrt{x-1}+2\cos\sqrt{x-1}+C.$
3. $-3\left(\sqrt[3]{x^2}-2\right)\cos\sqrt[3]{x^2}+6\sqrt[3]{x^2}\operatorname{sen}\sqrt[3]{x^2}+C.$
4. $\frac{1}{2}\left(x^2-\frac{1}{2}\right)\operatorname{arcsen}x+\frac{1}{4}x\sqrt{1-x^2}+C.$
5. $\frac{1}{3}x^3\arctan x-\frac{1}{6}x^2+\frac{1}{6}\ln\left(x^2+1\right)+C.$
6. $3\left[\sqrt[3]{(x+2)^2}-2\sqrt[3]{x+2}+2\right]e^{\sqrt[3]{x+2}}+C.$
7. $\frac{1}{3}x^3\operatorname{arcsec}x-\frac{1}{6}\left[x\sqrt{x^2-1}+\ln\left(x+\sqrt{x^2-1}\right)\right]+C.$
8. $(x+1)\arctan\sqrt{x}-\sqrt{x}+C.$
9. $\frac{3}{2}\left(x^{\frac{1}{3}}-2\right)^2+\ln\left[\frac{\left(x^{\frac{1}{3}}+1\right)^4}{\sqrt{x^{\frac{2}{3}}-x^{\frac{1}{3}}+1}}\right]+\sqrt{3}\arctan\left(\frac{2x^{\frac{1}{3}}-1}{\sqrt{3}}\right)+C.$
10. $\left(\sqrt{x+1}+3\right)^2+\frac{3}{5}\ln\left[\frac{\left(\sqrt{x+1}-2\right)^{16}}{2\sqrt{x+1}+1}\right]+K.$
11. $\ln\left[\frac{C(\operatorname{sen}x-1)}{\operatorname{sen}x+2}\right].$
12. $\ln\left[\frac{C\left(e^x-2\right)}{e^x-1}\right].$
13. $\ln\left[\frac{C\tan x}{\tan x+1}\right].$
14. $\ln\left(1+\sec^2 x\right)-2\arctan(\cos x)+K.$
15. $\ln\left[\frac{K\left(e^x-1\right)}{\sqrt{e^{2x}+1}}\right]+\arctan\left(e^x\right).$
16. $\frac{1}{2}x^2\arctan\left(\frac{1}{x+1}\right)+\frac{1}{2}(x+1)-\ln\sqrt{(x+1)^2+1}+C.$
17. $\frac{1}{2}x^2\arctan\left(\frac{1}{x^2+1}\right)+\frac{1}{2}\ln\sqrt{\left(x^2+1\right)^2+1}-\frac{1}{2}\arctan\left(x^2+1\right)+C.$
18. $\ln\left[\frac{e^{2x}-1}{e^{2x}+1}\right]+K.$
19. $\frac{1}{4}x-\frac{5}{6}\arctan\left(\frac{1}{3}\tan\frac{x}{2}\right)+K.$
20. $\ln\left[K\left(\tan\frac{x}{2}+1\right)\right].$
21. $-x-\frac{4}{\tan\frac{x}{2}+1}+K.$
22. $\frac{23}{25}\ln[-3\cos x-4\operatorname{sen}x]-\frac{14}{25}x+K.$