ECUACIONES DIFERENCIALES
ECUACIONES HOMOGÉNEAS E100

Obtener la solución general de las edos:

(1)

$$xy' = \sqrt{x^2 - y^2} + y$$

(2)

$$(x^2 + xy + 3y^2)\,dx - (x^2 + 2xy)\,dy = 0$$

(3)

$$3x - 4y + (2x - y)y' = 0$$

(4)

$$\frac{dy}{dx} = \frac{y + x\cos^2\left(\dfrac{y}{x}\right)}{x}; \quad y(1) = \frac{\pi}{4}$$

(5)

$$y\,dx + x(\ln x - \ln y - 1)\,dy = 0; \quad y(1) = e$$

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Respuestas

(1) Obtener la solución general de las edo:

$$xy' = \sqrt{x^2 - y^2} + y$$

Dividiendo entre x

$$\begin{aligned} \frac{dy}{dx} &= \frac{\sqrt{x^2 - y^2}}{x} + \frac{y}{x} = \\ &= \sqrt{\frac{x^2 - y^2}{x^2}} + \frac{y}{x} \\ &= \sqrt{\frac{x^2}{x^2} - \frac{y^2}{x^2}} + \frac{y}{x} \end{aligned}$$

(A)
$$\frac{dy}{dx} = \sqrt{1 - \left(\frac{y}{x}\right)} + \frac{y}{x}$$

Se efectua un cambio de variable

Si

$$\frac{y}{x} = w \Rightarrow y = xw$$

De donde

$$\frac{dy}{dx} = x\frac{dw}{dx} + w$$

Sustituyendo en (A)

$$\begin{aligned} x\frac{dw}{dx} + w &= \sqrt{1 - w^2} + w \\ x\frac{dw}{dx} &= \sqrt{1 - w^2} \end{aligned}$$

Separando variables

$$\frac{dw}{\sqrt{1 - w^2}} = \frac{dx}{x}$$

Integrando

$$\begin{aligned} \int \frac{dw}{\sqrt{1 - w^2}} &= \int \frac{dx}{x} \\ \operatorname{arcsen} w &= \ln x + c_1 = \ln x + \ln c \\ \operatorname{arcsen} w &= \ln(cx) \end{aligned}$$

De donde

$$w = \operatorname{sen}(\ln cx)$$

Pero $w = \dfrac{y}{x}$, entonces

$$\frac{y}{x} = \operatorname{sen}(\ln cx)$$

Por lo que

$$y = x \operatorname{sen}(\ln cx)$$

(2) Obtener la solución general de las edo:

$$(x^2 + xy + 3y^2)\, dx - (x^2 + 2xy)\, dy = 0$$

$$\begin{aligned}
(x^2 + 2xy)\, dy &= (x^2 + xy + 3y^2)\, dx \\
\frac{dy}{dx} &= \frac{x^2 + xy + 3y^2}{x^2 + 2xy} \\
&= \frac{x^2\left(1 + \frac{y}{x} + 3\frac{y^2}{x^2}\right)}{x^2\left(1 + 2\frac{y}{x}\right)}
\end{aligned}$$

(B) $$\frac{dy}{dx} = \frac{1 + \frac{y}{x} + 3\left(\frac{y}{x}\right)^2}{1 + 2\left(\frac{y}{x}\right)}$$

si

$$\frac{y}{x} = w \Rightarrow y = xw \Rightarrow \frac{dy}{dx} = x\frac{dw}{dx} + w$$

Sustituyendo en (B) se obtiene

$$\begin{aligned}
x\frac{dw}{dx} + w &= \frac{1 + w + 3w^2}{1 + 2w} \\
x\frac{dw}{dx} &= \frac{1 + w + 3w^2}{1 + 2w} - w = \frac{1 + w + 3w^2 - w - 2w^2}{1 + 2w} \\
x\frac{dw}{dx} &= \frac{w^2 + 1}{2w + 1}
\end{aligned}$$

Separando variables

$$\frac{2w + 1}{w^2 + 1}\, dw = \frac{dx}{x}$$

Integrando

$$\int \frac{2w}{w^2+1}\,dw + \int \frac{dw}{w^2+1} = \int \frac{dx}{x}$$
$$\ln(w^2+1) + \arctan w = \ln x + c$$

Pero

$$w = \frac{y}{x}$$

Entonces

$$\ln\left(\frac{y^2}{x^2}+1\right) + \arctan\frac{y}{x} = \ln x + c$$
$$\ln\left(\frac{y^2+x^2}{x^2}\right) - \ln x + \arctan\frac{y}{x} = c$$
$$\ln(y^2+x^2) - \ln x^2 - \ln x + \arctan\frac{y}{x} = c$$
$$\ln(x^2+y^2) - 3\ln x + \arctan\frac{y}{x} = c$$
$$\ln(x^2+y^2) - \ln x^3 + \arctan\frac{y}{x} = c$$
$$\ln\left(\frac{x^2+y^2}{x^3}\right) + \arctan\frac{y}{x} = c$$

(3) Obtener la solución general de las edo:

$$3x - 4y + (2x-y)y' = 0$$

$$(2x-y)\frac{dy}{dx} = 4y - 3x$$
$$\frac{dy}{dx} = \frac{4y-3x}{2x-y} =$$
$$= \frac{x\left(4\frac{y}{x}-3\right)}{x\left(2-\frac{y}{x}\right)}$$

(C) $$\frac{dy}{dx} = \frac{4\left(\frac{y}{x}\right)-3}{2-\left(\frac{y}{x}\right)}$$

Si

$$\frac{y}{x} = u \Rightarrow y = xu \Rightarrow \frac{dy}{dx} = x\frac{du}{dx} + u$$

Sustituyendo en (C) se tiene que

$$x\frac{du}{dx} + u = \frac{4u-3}{2-u}$$

De donde

$$\begin{aligned} x\frac{du}{dx} &= \frac{4u-3}{2-u} - u = \frac{4u-3-2u+u^2}{2-u} \\ x\frac{du}{dx} &= \frac{u^2+2u-3}{2-u} \end{aligned}$$

Separando variables

$$\frac{2-u}{u^2+2u-3}\,du = \frac{dx}{x}$$

Integrando (mediante fracciones parciales el primer miembro)

$$\begin{aligned} \int \frac{-u+2}{(u+3)(u-1)}\,du &= \int \frac{dx}{x} \\ -\frac{5}{4}\ln(u+3) + \frac{1}{4}(u-1) &= \ln x + c_1 \end{aligned}$$

Multiplicando por 4

$$\begin{aligned} -5\ln(u+3) + \ln(u-1) &= 4\ln x + c_2 \\ \ln(u-1) - \ln(u+3)^5 &= \ln x^4 + \ln c \\ \ln\left[\frac{u-1}{(u+3)^5}\right] &= \ln(cx^4) \Rightarrow \frac{u-1}{(u+3)^5} = cx^4 \\ \Rightarrow u-1 = cx^4(u+3)^5&;\text{ pero } u = \frac{y}{x} \end{aligned}$$

Entonces

$$\begin{aligned} \frac{y}{x} - 1 = cx^4\left(\frac{y}{x}+3\right)^5 &\Rightarrow \frac{y-x}{x} = \left(\frac{y+3x}{x}\right)^5 \\ \Rightarrow y-x = cx^5\frac{(y+3x)^5}{x^5} &\Rightarrow y-x = c(y+3x)^5 \end{aligned}$$

(4) Obtener la solución general de las edo:

$$\frac{dy}{dx} = \frac{y + x\cos^2\left(\frac{y}{x}\right)}{x}; \quad y(1) = \frac{\pi}{4}$$

(D)
$$\frac{dy}{dx} = \frac{y}{x} + \cos^2\left(\frac{y}{x}\right)$$

Si

$$\frac{y}{x} = w \Rightarrow y = wx \Rightarrow \frac{dy}{dx} = x\frac{dw}{dx} + w$$

Sustituyendo en (D) se obtiene

$$\begin{aligned} x\frac{dw}{dx} + w &= w + \cos^2 w \\ x\frac{dw}{dx} &= \cos^2 w \end{aligned}$$

Separando variables

$$\frac{dw}{\cos^2 w} = \frac{dx}{x}$$

Integrando

$$\int \sec^2 w \, dw = \int \frac{dx}{x} \Rightarrow \tan w = \ln x + c$$

Pero $w = \dfrac{y}{x}$, entonces

$$\tan\left(\frac{y}{x}\right) = \ln x + c$$

Considerando que $y(1) = \dfrac{\pi}{4}$ se tiene que

$$\tan\left(\frac{\pi}{4}\right) = \ln 1 + c$$

de donde $c = 1$, por lo tanto

$$\begin{aligned} \tan\left(\frac{y}{x}\right) &= \ln x + 1 = \ln x + \ln e = \ln(ex) \\ \Rightarrow \frac{y}{x} &= \arctan(\ln ex) \Rightarrow y = x \arctan(\ln ex) \end{aligned}$$

(5) Obtener la solución general de las edo:

$$y\,dx + x(\ln x - \ln y - 1)\,dy = 0; \quad y(1) = e$$

$$x(\ln x - \ln y - 1)dy = -y\,dx$$

$$\begin{aligned}
\frac{dy}{dx} &= \frac{y}{x(\ln x - \ln y - 1)} \\
&= \frac{\frac{y}{x}}{\ln y + 1 - \ln x} \\
&= \frac{\frac{y}{x}}{\ln y + \ln e - \ln x} \\
\frac{dy}{dx} &= \frac{\frac{y}{x}}{\ln \frac{ye}{x}} \\
\frac{dy}{dx} &= \frac{\frac{y}{x}}{\ln e\left(\frac{y}{x}\right)}
\end{aligned} \tag{E}$$

Si

$$\frac{y}{x} = u \Rightarrow y = xu \Rightarrow \frac{dy}{dx} = x\frac{du}{dx} + u$$

Sustituyendo en (E) se obtiene

$$\begin{aligned}
x\frac{du}{dx} + u &= \frac{u}{\ln eu} = \frac{u}{1 + \ln u} \\
x\frac{du}{dx} &= \frac{u}{1 + \ln u} - u = \frac{u - u - u\ln u}{1 + \ln u} \\
x\frac{du}{dx} &= -\frac{u\ln u}{1 + \ln u}
\end{aligned}$$

Separando variables

$$\frac{1 + \ln u}{u\ln u}\,du = -\frac{dx}{x}$$

Integrando

$$\int \frac{du}{u\ln u} + \int \frac{du}{u} = -\int \frac{dx}{x}$$
$$\ln(\ln u) + \ln u = -\ln x + c$$
$$\ln(\ln u) + \ln u + \ln x = c$$
$$\ln(xu\ln u) = c$$

Pero $u = \dfrac{y}{x}$, entonces

$$\ln\left[x\,\frac{y}{x}\ln\left(\frac{y}{x}\right)\right] = c \Rightarrow \ln\left[y\ln\frac{y}{x}\right] = c$$

Considerando que $y(1) = e$, se tiene que

$$\ln\left[e\;\ln\frac{e}{1}\right] = c \Rightarrow c = 1$$

Por lo que

$$\ln\left[y\;\ln\frac{y}{x}\right] = 1$$

De donde

$$y\ln\frac{y}{x} = e$$